Elementary Discussions on Group Theory: A Physicists' Point of View

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Contents

This note is based upon the following excellent textbooks:

`Tinkham: Group Theory & Quantum Mechanics`
`Joshi: Elements of Group Theory for Physicists`
`Georgi: Lie Groups in Particle Physics`
`Cheng and Li: Gauge Theory of Elementary Particle Physics`
`Tung: Group Theory in Physics`
`Schweber: Relativistic Quantum Field Theory`

You are always advised to read the original textbooks. Remember that the supplementary problems form an integral part of the course.

Caveat Emptor: This note emerged from the lectures I have delivered over the last few years. The coverage is far from exhaustive. In fact, there are a number of important topics (*e.g.*, rotation group) that I generally do not have time to touch in the class, and have incorporated here only briefly. Maybe I will try to do full justice to them later. The language may not be as clear as I have planned. I will be extremely glad to have suggestions. The figures have been drawn with the software `xfig`, and I have not given too much time for a flawless job there. Be careful!

1 Why Should We Study Group Theory?

Group theory can be developed, and was developed, as an abstract mathematical topic. However, we are not mathematicians. We plan to use group theory only as much as is needed for physics purpose. For this, we focus more on physics aspects than on mathematical rigour. All complicated proofs have been carefully avoided, but you should consult the reference books if you are interested.

Almost every time, we have to use some symmetry property of the system under study to get more information (like the equations of motion, or the energy eigenfunctions) about it. For example, if the potential in the Schrödinger equation is symmetric under the exchange $\mathbf{x} \to -\mathbf{x}$ (this is known as a parity transformation), even without solving, we can say that the wavefunctions are bound to have a definite parity. Group theory is nothing but a mathematical way to study such symmetries. The symmetry can be discrete (*e.g.*, reflection about some axis) or continuous (*e.g.*, rotation). Thus, we need to study both discrete and continuous groups. The former is used more in solid state physics, particularly in crystallographic studies, while the latter is used exhaustively in quantum mechanics, quantum field theory, and nuclear and particle physics.

2 What is a Group?

A *group* G is a set of discrete elements $a, b, \cdots x$ alongwith a group operator [1], which we will denote by $\odot$, with the following properties:

- *Closure*: For any two elements a, b in G, $a \odot b$ must also be in G. We will try to avoid the mathematical symbols as far as practicable, but let me tell you that in some texts this is written as $\forall a, b \in G, a \odot b \in G$. The symbol $\forall$ is a shorthand for "for all". Another commonly used symbol is $\exists$: $\exists a \in G$ means "there exists an a in G such that".

- *Associativity*: For any three elements a, b, c in G,

$$a \odot (b \odot c) = (a \odot b) \odot c, \tag{1}$$

  i.e., the order of the operation is not important.

- *Identity*: The set must contain an identity element e for which $a \odot e = e \odot a = a$.

- *Inverse*: For every a in G, there must be an element $b \equiv a^{-1}$ in G so that $b \odot a = a \odot b = e$ (we do not distinguish between left and right inverses). The inverse of any element a is unique; prove it.

These are essential properties of a group. Furthermore, if $a \odot b = b \odot a$, the group is said to be *abelian*. If not, the group is *non-abelian*. Note that no two elements of a group are identical. If the number of group elements (this is also called the *order* of the group) is finite, it is a *finite group*; otherwise it is an *infinite group*. Elements of a finite group are necessarily

[1] In most of the texts this is called *group multiplication* or simply *multiplication* operator. Let me warn you that the operator can very well be something completely different from ordinary or matrix multiplication. For example, it can be addition for the group of all integers.

discrete (finite number of elements between any two elements). An infinite group may have discrete or continuous elements.

Now some examples:

- The additive group of all integers. It is an abelian infinite group. The group operation is addition, the identity is 0, and the inverse of a is just $-a$. Note that the set of all positive integers is not a group; there is no identity or inverse.

- The multiplicative group of all real numbers excluding zero (why?).

- Z_n, the multiplicative group of n-th roots of unity (this is also called the cyclic group of order n). For example, $Z_3 = \{1, \omega, \omega^2\}$ where $\omega = 1^{1/3}$; $Z_4 = \{1, i, -1, -i\}$. The operation is multiplication and the identity is 1 (what is the inverse?).

- The n-object permutation group S_n. Consider a 3-element permutation group $S_3 = \{a, b, c\}$. There are six elements: P_0, which is the identity and does not change the position of the elements; P_{12}, which interchanges positions 1 and 2, *i.e.*, $P_{12} \to (b, a, c)$. Similarly, there are P_{13} and P_{23}. Finally, there are P_{123} and P_{132}, with

$$\begin{aligned} P_{123}(a, b, c) &\to (c, a, b) \\ P_{132}(a, b, c) &\to (b, c, a), \end{aligned} \tag{2}$$

  i.e., P_{123} takes element of position 1 to position 2, that of position 2 to position 3 and that of position 3 to position 1. Obviously, $P_{123} = P_{231} = P_{312}$, and similarly for P_{132}.

- C_{4v}, the symmetry group of a square. Consider a square in the x-y plane with corners at (a, a), $(a, -a)$, $(-a, -a)$, and $(-a, a)$. The symmetry operations are rotations about the z axis by angles $\pi/2$, π and $3\pi/2$ (generally, they are taken to be anticlockwise, but one can take clockwise rotations too), reflections about x and y axes, and about the diagonals.

- $U(n)$, the group of all $n \times n$ unitary matrices. Thus, the elements of $U(1)$ are pure phases like $\exp(i\theta)$. It is the group of phase transformations. Apart from $U(1)$, all $U(n)$s are non-abelian. When we say that an individual phase in the wave function does not have any physical significance, we mean that the Lagrangian is so constructed that it is invariant under a $U(1)$ transformation $\psi \to e^{i\theta}\psi$.

- $SU(n)$, the group of all $n \times n$ unitary matrices with determinant unity (also called unitary unimodular matrices). This is the most important group in particle physics. All SU groups are non-abelian, of which the simplest is $SU(2)$. The simplest member of $SU(2)$ is the two-dimensional rotation matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, characterised by different values of θ. Check that the matrix $\begin{pmatrix} e^{i\alpha}\cos\theta & e^{i\beta}\sin\theta \\ -e^{-i\beta}\sin\theta & e^{-i\alpha}\cos\theta \end{pmatrix}$ is also a member of $SU(2)$.

- $O(n)$, the group of all $n \times n$ orthogonal matrices. It is non-abelian for $n \geq 2$.

- $SO(n)$, the group of all $n \times n$ orthogonal matrices with determinant unity. Thus, $SO(3)$ is the familiar rotation group in three dimensions. The S in $SU(n)$ and $SO(n)$ stands for *special*, *viz.*, the unimodularity property.

The first group is infinite but *discrete*, *i.e.*, there are only a finite number of group elements between any two elements of the group. The second group is infinite and continuous, since there are infinite number of reals between any two real numbers, however close they might be. Z_n and S_n are obviously discrete, since they are finite. The last four groups are infinite and *continuous*, which means that there are an infinite number of group elements between two given elements of the group. We will be interested in only those continuous groups whose elements can be parametrised by a finite number of parameters. Later, we will see that this number is equal to the number of generators of the group.

Q. Show that the number of independent elements of an $N \times N$ unitary matrix is N^2, and that of an $N \times N$ unimodular matrix is $N^2 - 1$. [Hint: $U^\dagger U = 1$, so det U det $U^\dagger = 1$, or $|\det U|^2 = 1$, so that the determinant must have modulus unity and a form like $\exp(i\theta)$.]
Q. Show that the determinant of a 2×2 orthogonal matrix must be either $+1$ or -1. *[Hint: Write the matrix as* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and construct the constraint equations. Show that this leads to* $(ad - bc)^2 = 1$.*]*

3 Discrete and Finite Groups

3.1 Multiplication Table

A multiplication table is nothing but a compact way to show the results of all possible compositions among the group elements. Obviously, this makes sense only for finite groups. For example, the multiplication tables for Z_4 and S_3 are, respectively,

	1	i	−1	−i
1	1	i	−1	−i
i	i	−1	−i	1
−1	−1	−i	1	i
−i	−i	1	i	−1

	P_0	P_{12}	P_{13}	P_{23}	P_{123}	P_{132}
P_0	P_0	P_{12}	P_{13}	P_{23}	P_{123}	P_{132}
P_{12}	P_{12}	P_0	P_{132}	P_{123}	P_{23}	P_{13}
P_{13}	P_{13}	P_{123}	P_0	P_{132}	P_{12}	P_{23}
P_{23}	P_{23}	P_{132}	P_{123}	P_0	P_{13}	P_{12}
P_{123}	P_{123}	P_{13}	P_{23}	P_{12}	P_{132}	P_0
P_{132}	P_{132}	P_{23}	P_{12}	P_{13}	P_0	P_{123}

One should note a few things. First, Z_4 is obviously abelian, and hence the multiplication table is symmetric, but S_3 is non-abelian (*e.g.*, $P_{13}P_{23} \neq P_{23}P_{13}$). For both of them, all elements occur *only once* in each row or column. This is a general property of the multiplication table and is easy to prove. Suppose two elements $a \odot b$ and $a \odot c$ are same. Multiply by a^{-1}, so $b = c$, contrary to our assumption of all elements being distinct. However, number of elements in each row or column is equal to the order of the group, so all elements must occur once and only once.

3.2 Isomorphism and Homomorphism

Consider the symmetry group of an equilateral triangle, with six elements (identity, three reflections about the medians, and rotations by $2\pi/3$ and $4\pi/3$). The multiplication table is identical with S_3. Only this (not just the number of elements) shows that the groups behave in an identical way. This is known as *isomorphism*: we say that these two groups are isomorphic to one another. Thus, isomorphism means a one-to-one correspondence between the elements of two groups so that if $a, b, c \in G$ and $d, e, f \in H$, and a, b, c are isomorphic to d, e, f respectively, then $a \odot b = c$ implies $d \odot e = f$, for all a, b, c and d, e, f. Note that e is just an element of H, not the identity; also, the operations in G and H may be completely different. Try to convince yourself that the identity of one group must be mapped on to the identity of the second group.

Thus, S_3 is isomorphic to C_3. If we consider only rotational symmetries, then the three-member subgroup (a subset of a group that itself behaves likes a group) of C_3 is isomorphic to Z_3. In fact, this is a general property: the rotational symmetry group of any symmetric n-sided polygon, having $2\pi/n$ rotation as a symmetry operation, is isomorphic to Z_n. Check this for Z_4. This cannot be a coincidence; what is the physical reason behind this?

If the mapping is not one-to-one but many-to-one, the groups are said to be homomorphic to one another. Obviously, in a many-to-one mapping, some information is lost. All groups are homomorphic to the group containing the identity; but that is a very bad mapping, since no information about the group structure is retained. A better homomorphism occurs between Z_2 and Z_4, where $(1, -1)$ of Z_4 is mapped to 1 of Z_2, and $(i, -i)$ of Z_4 is mapped to -1 of Z_2. Isomorphism is only a special case of homomorphism, but in general, the two groups which are homomorphic to one another should be of different order, and the ratio of their orders n/m should be an integer k. In this case, set of k elements of G is mapped to one element of H. The set containing identity in G must be mapped to the identity of H: prove this. Also prove that this set itself must form a group. Such a group, which is completely embedded in a bigger group, is known as a *subgroup*. Every group has two trivial subgroups: the identity and the entire group!

3.3 Conjugacy Classes

Consider a group G with elements $\{a, b, c, d, \cdots\}$. If $aba^{-1} = c$, (from now on we drop the $\odot$ symbol for group operation), then b and c are said to be conjugate elements. If b is conjugate to both c and d, then c and d are conjugate to each other. The proof goes like this: Suppose $aba^{-1} = c$ and $hbh^{-1} = d$, then $b = a^{-1}ca$ and hence $ha^{-1}cah^{-1} = d$. But ah^{-1} is a member of the group, and hence its inverse, ha^{-1}, too (why the inverse of ah^{-1} is not $a^{-1}h$?). So c and d are conjugate to each other.

It is trivial to show that the identity element in any group is conjugate only to itself, and for an abelian group all members are conjugate to themselves only.

Now, any discrete group can be separated into sets of elements (need not having the same number of elements) where all members of a set are conjugate to each other but no member of any set is conjugate to another member of a different set. In that case, the sets are called *conjugacy classes* or simply *classes*.

Let us take C_{4v}, the symmetry group of a square. The symmetry operations are 1 (identity), $r_{\pi/2}, r_\pi, r_{3\pi/2}$ (rotations, may be clockwise or anticlockwise), R_x, R_y (reflections

about x and y axes, passing through the centre of the square), and R_{NE} and R_{SE}, the reflections about the NE and the SE diagonal. [In some texts you will see different notations, but these are equally good, if not more transparent.]

These eight elements form a group; that can be checked from the multiplication table. The group is non-abelian. However, the first four members form an abelian group isomorphic to Z_4. The eight elements can be divided into five classes: (1), (r_π), $(r_{\pi/2}, r_{3\pi/2})$, (R_x, R_y), (R_{NE}, R_{SE}). It is left as an exercise to check the class structure.

What is the physical significance of classes? In other words, can we guess which elements should be in a particular class? The answer is yes: note that the conjugacy operation is nothing but a similarity transformation performed with the group elements. This will be more obvious in the next section when we show how to represent the abstract group elements with matrices, in particular unitary matrices.

Identity must be a class by itself. $r_{\pi/2}$ and $r_{3\pi/2}$ belong to the same class *because* there is an element of the group which relates rotation by $\pi/2$ with rotation by $3\pi/2$: just the reflection about the x or y axis, which makes a clockwise rotation anticlockwise and vice versa. Similarly, R_x and R_y belong to the same class since there is an operation that relates them: rotation by $\pi/2$. However, R_x and R_{NE} cannot belong to the same class, since there is no symmetry operation of $r_{\pi/4}$.

4 Representation of a Group

The permutation group discussed above is an example of a *transformation group* on a physical system. In quantum mechanics, a transformation of the system is associated with a unitary operator in the Hilbert space (time reversal is the only example of antiunitary transformation that is relevant to us). Thus, a transformation group of a quantum mechanical system is associated with a mapping of the group to a set of unitary operators [2]. Thus, for each a in G there is a unitary operator $D(a)$. Furthermore, this mapping must preserve the group operation, *i.e.*,

$$D(a)D(b) = D(a \odot b) \tag{3}$$

for all a and b in G. A mapping which satisfies eq. (3) is called a *representation* of the group G. In fact, a representation can involve nonunitary operators so long as they satisfy eq. (3).

For example, the mapping

$$D(n) = \exp(in\theta) \tag{4}$$

is a representation of the additive group of integers, since

$$\exp(im\theta)\exp(in\theta) = \exp(i(n+m)\theta). \tag{5}$$

Check that the following mapping is a representation of the 3-element permutation group S_3:

$$D(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ D(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ D(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

[2] If this mapping is one-to-one, the *representation* is called *faithful*. We will deal with faithful representations only.

$$D(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; \; D(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} ; \; D(321) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} . \tag{6}$$

Particularly, check the multiplication table.

In short, a representation is a specific realisation of the group operation law by finite or infinite dimensional matrices. For abelian groups, their representative matrices commute.

Consider a n-dimensional Hilbert space. Thus, there are n orthonormal basis vectors. Let $|i\rangle$ be a normalised basis vector. We define the ij-th element of any representation matrix $D(a)$ as

$$[D(a)]_{ij} = \langle i|D(a)|j\rangle \,, \tag{7}$$

so that

$$D(a)|i\rangle = \sum_j |j\rangle\langle j|D(a)|i\rangle = \sum_j [D(a)]_{ji}|j\rangle \,. \tag{8}$$

From now on, we will freely translate from one language (representations as abstract linear operators) to the other (representations as matrices). Anyway, it is clear that the dimension of the representation matrices is the same as that of the Hilbert space.

Thus, the elements of a group can be represented by matrices. The dimension of the matrices has nothing to do with the order of the group. But the matrices should be all different, and there should be a one-to-one mapping between the group elements and the matrices; that's what we call a faithful representation. Obviously, the representative matrices are square, because matrix multiplication is defined both for $D(a)D(b)$ and $D(b)D(a)$. The representation of identity must be the unit matrix, and if $T(a)$ is the representation of a, then $T^{-1}(a) = T(a^{-1})$ is the representation of a^{-1}. The matrices must be non-singular; the inverse exists.

However, the matrices need not be unitary. But in quantum mechanics we will be concerned with unitary operators, and so it is better to show now that any representation of a group is *equivalent* to a representation by unitary matrices. Two representations D_1 and D_2 are *equivalent* if they are related by a *similarity transformation*

$$D_2(a) = SD_1(a)S^{-1} \tag{9}$$

with a fixed operator S for all a in the group G.

Suppose $T(a)$ is the representation of a group G. $T(a)$s need not be unitary. Define a hermitian matrix

$$H = \sum_{a\in G} T(a)T^\dagger(a). \tag{10}$$

This matrix, being hermitian, can be diagonalised by a unitary transformation. Let $U^\dagger HU = H_d$. H_d is a real diagonal matrix whose elements are the eigenvalues of H. Using the form of H,

$$H_d = U^\dagger \sum_{a\in G} T(a)T^\dagger(a)U = \sum_{a\in G} \left(U^\dagger T(a)U\right)\left(U^\dagger T^\dagger(a)U\right) = \sum_{a\in G} \mathcal{T}(a)\mathcal{T}^\dagger(a) \tag{11}$$

where $\mathcal{T}(a)$ is also a representation (remember $U^\dagger = U^{-1}$). Take the k-th diagonal element of H_d:

$$[H_d]_{kk} \equiv d_k = \sum_{a\in G}\sum_j \mathcal{T}_{kj}(a)\mathcal{T}^\dagger_{jk}(a) = \sum_{a\in G}\sum_j |\mathcal{T}_{kj}(a)|^2. \tag{12}$$

Thus $d_k \geq 0$. The case $d_k = 0$ can be ruled out since in that case a particular row in all the representative matrices is zero and the determinants are zero, so all matrices are singular. So d_k is positive for all k, and we can define a diagonal matrix $H_d^{1/2}$ whose k-th element is $\sqrt{d_k}$. Construct the matrix $V = UH_d^{1/2}$ and the representation $\Gamma(a) = V^{-1}T(a)V$. We now show that all Γ matrices are unitary, completing the proof.

We have $\Gamma(a) = V^{-1}T(a)V = H_d^{-1/2}U^{-1}T(a)UH_d^{1/2} = H_d^{-1/2}\mathcal{T}(a)H_d^{1/2}$, and

$$\begin{aligned} \Gamma(a)\Gamma^\dagger(a) &= \left[H_d^{-1/2}\mathcal{T}(a)H_d^{1/2}\right]\left[H_d^{1/2}\mathcal{T}^\dagger(a)H_d^{-1/2}\right] \\ &= H_d^{-1/2}\mathcal{T}(a)H_d\mathcal{T}^\dagger(a)H_d^{-1/2} \\ &= H_d^{-1/2}\mathcal{T}(a)\sum_{b\in G}\mathcal{T}(b)\mathcal{T}^\dagger(b)\mathcal{T}^\dagger(a)H_d^{-1/2} \\ &= H_d^{-1/2}\sum_{b\in G}\mathcal{T}(ab)\mathcal{T}^\dagger(ab)H_d^{-1/2} \\ &= H_d^{-1/2}H_dH_d^{-1/2} = \mathbf{1}. \end{aligned} \tag{13}$$

Here we have used the rearrangement theorem: as long as we sum over all the elements of the group, how we denote them is immaterial.

This theorem depends on the convergence of a number of sums. For an infinite group, this is not so straightforward. However, for Lie groups (to be discussed in the next section) this theorem holds.

4.1 Reducible and Irreducible Representations

A representation D is *reducible* if it is equivalent to a representation D' with *block-diagonal* form (or itself block-diagonal):

$$D'(x) = SD(x)S^{-1} = \begin{pmatrix} D_1'(x) & 0 \\ 0 & D_2'(x) \end{pmatrix}. \tag{14}$$

The vector space on which D' acts breaks up into two orthogonal subspaces, each of which is mapped into itself by all the operators $D'(x)$. The representation D' is said to be the *direct sum* of D_1' and D_2':

$$D' = D_1' \oplus D_2'. \tag{15}$$

A representation is *irreducible* if it cannot be put into block-diagonal form by a similarity transformation.

4.2 Schur's Lemma and the Great Orthogonality Theorem

Before closing the section on discrete groups, we will prove a very important theorem on the orthogonality of different representations of a group. This theorem tells you that if you have two different representations of a group which are both irreducible but inequivalent to each other, then the 'dot product' of these two representations is zero. What is a 'dot product'? Each of these representations form a g-dimensional vector space, where g is the order of the group. The 'dot product' is something like taking a matrix from one space, taking its corresponding matrix from the other space, taking any two elements of these matrices, and

then sum over all elements of the group. A more mathematical definition will soon follow, but before that we need to prove two lemmas by Schur.

Schur's Lemma 1. If a matrix P commutes with all representative matrices of an irreducible representation, then $P = c\mathbf{1}$, a multiple of the unit matrix.

Proof. Given, $PT(a) = T(a)P$ for all $a \in G$, where we take $T(a)$s to be a unitary representation without loss of generality. Suppose the dimension of $T(a)$ is $n \times n$; evidently, that should be the dimension of P. $T(a)$s possess a complete set of n eigenvectors. So does P, since $[P, T(a)] = 0$. Let ψ_j be some eigenvector of P with eigenvalue c_j: $P\psi_j = c_j\psi_j$. Then $PT(a)\psi_j = c_jT(a)\psi_j$. So both $\psi_j = T(1)\psi_j$ and $T(a)\psi_j$, two independent eigenvectors, have same eigenvalue. How many such degenerate states exist? Obviously n, since if the number of such degenerate states be $m < n$, we have an m-dimensional invariant subspace in the whole n-dimensional vector space, so the original space is not irreducible. Thus, all c_js are equal, and $P = c\mathbf{1}$.

Schur's Lemma 2. Suppose you have two irreducible representations $T^i(a)$ and $T^j(a)$ of dimension l_i and l_j respectively. If a matrix M satisfies $T^i(a)M = MT^j(a)$ for all $a \in G$, then either (i) $M = 0$, a null matrix, or (ii) *det* $M \neq 0$, in which case T^i and T^j are equivalent representations.

Proof. The dimension of M is $l_i \times l_j$. If $l_i = l_j$, M is a square matrix. If its determinant is not zero, T^i and T^j are obviously equivalent representations.

First, we show, with the help of the first lemma, that $M^\dagger M$ is a multiple of the unit matrix. Take the hermitian conjugate of both sides of the defining equation:

$$\begin{aligned} M^\dagger {T^i}^\dagger(a) = {T^j}^\dagger(a)M^\dagger \quad &\Rightarrow \quad M^\dagger T^i(a^{-1}) = T^j(a^{-1})M^\dagger, \\ M^\dagger T^i(a^{-1})M = T^j(a^{-1})M^\dagger M \quad &\Rightarrow \quad M^\dagger M T^j(a^{-1}) = T^j(a^{-1})M^\dagger M. \end{aligned} \tag{16}$$

This is true for all the elements of G, so $M^\dagger M = c\mathbf{1}$. If M is a square matrix and $c \neq 0$, then the representations are equivalent. If $c = 0$, then

$$(M^\dagger M)_{ii} = \sum_k |M_{ki}|^2 = 0. \tag{17}$$

This is true only if $M_{ki} = 0$ for all k. But i is arbitrary and can run from 1 to n. So $M = 0$.

Now suppose $l_i \neq l_j$. Let $l_i < l_j$. Add $l_j - l_i$ rows of zero to M to get a square matrix $\mathcal{M}$, whose determinant is obviously zero. But $\mathcal{M}^\dagger\mathcal{M} = M^\dagger M$, and since the first one is zero, so is the second one. Again take the (i, i)-th element of $M^\dagger M$ to get $M = 0$.

The Great Orthogonality Theorem says that if T^i and T^j are two inequivalent irreducible representations of a group G, then

$$\sum_{a \in G} T^i_{km}(a)T^j_{ns}(a^{-1}) = \frac{g}{l_i}\delta_{ij}\delta_{ks}\delta_{mn}, \tag{18}$$

where l_i and l_j are the dimensions of the representations T^i and T^j respectively, and g is the order of the group.

Proof. Consider a matrix M constructed as

$$M = \sum_{a \in G} T^i(a)XT^j(a^{-1}) \tag{19}$$

where X is an arbitrary matrix, independent of the group elements. Note that if the representations are unitary then $T^j(a^{-1}) = T^{j\dagger}(a)$. Let T^i and T^j be two inequivalent irreducible

representations of dimensions l_i and l_j respectively. Multiplying both sides of eq. (19) by $T^i(b)$, where $b \in G$, we get

$$\begin{aligned} T^i(b)M &= \sum_{a\in G} T^i(ba)XT^j(a^{-1}) = \sum_{c\in G} T^i(c)XT^j(c^{-1}b) \\ &= \sum_{c\in G} T^i(c)XT^j(c^{-1})T^j(b) = MT^j(b), \end{aligned} \tag{20}$$

and so, by the second lemma, $M = 0$. But this is true for any X; let $X_{pq} = \delta_{pm}\delta_{qn}$, *i.e.*, only the mn-th element is 1 and rest 0. Take the ks-th element of M:

$$M_{ks} = \sum_{a\in G}\sum_{p,q} T^i_{kp}(a)X_{pq}T^j_{qs}(a^{-1}) = 0. \tag{21}$$

But this reduces to

$$\sum_{a\in G} T^i_{km}(a)T^j_{ns}(a^{-1}) = 0. \tag{22}$$

Next, construct a matrix $N = \sum_{a\in G} T^i(a)XT^i(a^{-1})$. By an argument similar to eq. (20), we get $NT^i(a) = T^i(a)N$ for all $a \in G$, so N is a multiple of the unit matrix: $N = c\mathbf{1}$. Taking the ks-th element of N, and the same form of X, we find

$$\sum_{a\in G} = T^i_{km}(a)T^i_{ns}(a^{-1}) = c\delta_{ks}. \tag{23}$$

To get c, take the trace of N, which is a multiple of an $l_i \times l_i$ dimensional unit matrix:

$$\begin{aligned} Tr(N) = cl_i &= \sum_{a\in G}\sum_{k,p,q} T^i_{kp}(a)X_{pq}T^i_{qk}(a^{-1}) \\ &= \sum_{p,q} X_{pq}\sum_{a\in G}\sum_{k} T^i_{qk}(a^{-1})T^i_{kp}(a) \\ &= \sum_{p,q} X_{pq}\sum_{a\in G} T^i_{qp}(1) = g\sum_{p,q} X_{pq}\delta_{pq} = g\ Tr(X), \end{aligned} \tag{24}$$

so that $c = g\ Tr(X)/l_i$. But $Tr(X) = 0$ if $m \neq n$, or $Tr(X) = \delta_{mn}$. Combining this with eq. (22), we get

$$\sum_{a\in G} T^i_{km}(a)T^j_{ns}(a^{-1}) = \frac{g}{l_i}\delta_{ij}\delta_{ks}\delta_{mn}. \tag{25}$$

5 Lie Groups and Lie Algebras

A *Lie group* is, for our purpose, a group whose elements are labeled by a set of continuous parameters with a group operation law that depends smoothly on the parameters. This is also known as a *continuous connection to identity.* We will be interested in *compact Lie group*s; in a certain sense, the volume of the parameter space of a compact group is finite. For example, translation is not a compact group, since you can go upto infinity; but rotation is, since after a certain time you are bound to come back to the same point.

Any representation of a compact Lie group is *equivalent to a representation by unitary operators* (we have proved an analogous statement for discrete groups earlier). Any group element which can be obtained from the identity by continuous changes of infinitesimal real parameters ϵ_a can be written as

$$lim_{n\to\infty}(1 + i\epsilon_a X_a)^n \to \exp(i\alpha_a X_a) \tag{26}$$

where a, which runs from 1 to N, has been summed over, and $\alpha = n\epsilon$ is a finite parameter. The linearly independent hermitian operators X_a, which are called *group generators*, form a basis in the space of all linear combinations $\alpha_a X_a$ [3]. The dimension of this space is N; this is the space of the group generators, and not to be confused with the Hilbert space on which the generators act, which may have a completely different dimension. Thus, a group may have different unitary representations, but the number of generators must be the same.

Consider a rotation in two dimensions. An infinitesimal rotation by θ takes (x, y) to $(x + y\theta,\ -x\theta + y)$, so that the only generator of this group is

$$X = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{27}$$

i.e., the Pauli matrix σ_2. Applying this infinitesimal transformation a number of times, we can generate any rotation, *i.e.*, any element of the 2-dimensional rotation group. We say that all such elements are *continuously connected to identity*, and can be expressed as $\exp(i\theta\sigma_2)$.

The generators, as we have already said, form a vector space; thus, any linear combination of them is a generator. They also satisfy simple commutation relations which determine almost the full structure of the group, apart from some numerical normalisation. Consider the product

$$\exp(i\lambda X_b)\exp(i\lambda X_a)\exp(-i\lambda X_b)\exp(-i\lambda X_a) = 1 + \lambda^2[X_a, X_b] + \cdots \tag{28}$$

Because of the group property, the product of a number of group elements is another group element; it can be written as $\exp(i\beta_c X_c)$. As $\lambda \to 0$ we must have

$$\lambda^2[X_a, X_b] = i\beta_c X_c \tag{29}$$

or, writing $\beta_c/\lambda^2 \equiv f_{abc}$,

$$[X_a, X_b] = i f_{abc} X_c. \tag{30}$$

Note that a sum over c is implied; unless otherwise stated explicitly, we will always use the summation convention. The real constants f_{abc} are called the *structure constants* of the group. They are completely determined by the group operation law.

5.1 Algebra of Generators

The generators satisfy the *Jacobi identity*

$$[X_a, [X_b, X_c]] + \text{cyclic permutations} = 0. \tag{31}$$

It is obvious for a representation if we just write out the linear operators explicitly; it is also true for the abstract group generators. In terms of the structure constants, eq. (31) becomes

$$f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0. \tag{32}$$

(Prove eq. (32).)

If we define a set of matrices T_a by

$$(T_a)_{bc} \equiv -i f_{abc} \tag{33}$$

[3] That these operators, as well as their representative matrices, must be hermitian can be seen form the fact that the group matrices are unitary and α parameters are real.

then eq. (32) can be written as (note that the structure constants are antisymmetric in at least its first two indices; actually, as it will be shown later, it is completely antisymmetric in all its indices)

$$[T_a, T_b] = i f_{abc} T_c. \tag{34}$$

In other words, the structure constants themselves generate a representation of the group; this is called the *adjoint representation*. If the number of generators be N, the dimension of the adjoint representation matrices must be $N \times N$.

The generators and the commutation relations define the *Lie algebra* associated with the Lie group. Every representation of the group defines a representation of the algebra. The generators in the representation, when exponentiated, give the operators of the group representation. The definitions of equivalence, reducibility and irreducibility are the same for the algebra as for the group.

The structure constants depend on what basis we choose for the generators. To choose an appropriate basis, we use the adjoint representation defined by eq. (33). Consider the trace $Tr(T_a T_b)$. This is a real symmetric matrix, so we can diagonalise it by choosing appropriate real linear combinations of the X_as and therefore of the T_as. Suppose we do that, and get

$$Tr(T_a T_b) = k_a \delta_{ab} \quad \text{(no sum over } a\text{)}. \tag{35}$$

We still have the freedom to rescale the generators, so we could choose all the nonzero k_as to have the absolute value 1. However, we cannot change the sign of k_as.

In our future discussions, we will be concerned with algebras for which all k_as are positive. Thus we write, with suitable normalisation to all generators,

$$Tr(T_a T_b) = \lambda \delta_{ab} \tag{36}$$

for any convenient positive λ. We multiply both sides of eq. (34) by $\lambda^{-1} T_c$ and take trace to find

$$f_{abc} = -i \lambda^{-1} Tr([T_a, T_b], T_c) \tag{37}$$

so that f_{abc} is completely antisymmetric because of the cyclic property of the trace. Here T_as are hermitian; in fact, for compact Lie groups any representation is equivalent to a representation by hermitian operators and all irreducible representations are finite hermitian matrices. Thus, f_{abc} is always antisymmetric no matter what representation we choose.

An *invariant subalgebra* is some set of generators of a group G which goes either into itself or zero under commutation with any element of the whole algebra. Thus, if X is any generator of the invariant subalgebra and Y is any generator of the whole algebra, $[X, Y]$ is a generator of the invariant subalgebra (or it is zero). An algebra without any nontrivial invariant subalgebra (the whole and the null set are trivial) is called *simple*. A simple algebra generates a simple group.

One must take note of the *abelian invariant subalgebras*. The generators of an abelian group commute with everything; each of these generators is associated with what we call a $U(1)$ factor of the group. At any rate, $U(1)$ factors do not show up in the structure constants, because the structure constants of an abelian group are all zero. Algebras without any abelian invariant subalgebras are called *semisimple*.

It is the semisimple Lie groups which play a crucial role in particle physics. It can be trivially shown that any $U(n)$ ($n \neq 1$) group has one $U(1)$ factor:

$$U(n) \to SU(n) \times U(1). \tag{38}$$

$SU(n)$ groups are semisimple. The reason why we deal with $SU(n)$s rather than $U(n)$s is that an arbitrary phase in a wavefunction does not play any role in quantum mechanics (unless we are interested in interference phenomena, but that is a different story).

The generators, like the linear operators of the representations they generate, can be thought of as either linear operators or matrices:

$$X_a|i\rangle = |j\rangle[X_a]_{ji}. \tag{39}$$

In the Hilbert space, the group element $\exp(i\alpha_a X_a)$ transforms a ket $|i\rangle$ to $\exp(i\alpha_a X_a)|i\rangle$; thus, the corresponding bra will transform like $\langle i| \to \langle i|\exp(-i\alpha_a X_a)$. Any operator $\mathcal{O}$ transforms as

$$\mathcal{O} \to \exp(i\alpha_a X_a)\mathcal{O}\exp(-i\alpha_a X_a). \tag{40}$$

From the commutation relation of the generators (eq. (30)), it is clear that if X_as ($a = 1$ to N) are the generators of a particular group, so also are $-X_a^*$s (take the complex conjugate of both sides). If one can find some similarity transformation for which

$$SX_aS^{-1} = -X_a^* \tag{41}$$

for all a, (*i.e.*, if they are equivalent) X_as are said to construct a *real representation* of the group; if not, they construct a *complex representation*. All irreducible representations of $SU(2)$ are real (reducible representations are just direct sum of irreducible representations). This is not true for other $SU(n)$ groups; however, the adjoint representations are always real.

Given two representations D_1 (dim n) and D_2 (dim m) of a group G, we can form another representation in two ways. One is to have a direct sum of dimension $n+m$. This is a block-diagonal matrix, which reduces to two irreducible dim-n and dim-m representations. The generators of the representation $D_1 \oplus D_2$ are

$$X_a^{D_1\oplus D_2} = \begin{pmatrix} X_a^{D_1} & 0 \\ 0 & X_a^{D_2} \end{pmatrix}. \tag{42}$$

This operation is just the reverse of the process of reducing a representation.

We can also form a $n \times m$ dimensional representation as follows. If $|i\rangle$ ($i = 1$ to n) is an orthonormal basis in the space on which D_1 acts and $|x\rangle$ ($x = 1$ to m) is an orthonormal basis for D_2, we can identify the orthonormal product basis $|i\rangle|x\rangle$ in an $n \times m$ dimensional space. This is called the *direct product space*. On this space, the *direct product representation* $D_1 \otimes D_2$ is

$$(D_1 \otimes D_2)(a)\{|i\rangle|x\rangle\} = D_1(a)|i\rangle D_2(a)|x\rangle. \tag{43}$$

In the matrix language

$$[(D_1 \otimes D_2)(a)]_{ix,jy} = [D_1(a)]_{ij}[D_2(a)]_{xy}. \tag{44}$$

The generators of the direct product representation are the sums

$$[X_a^{D_1\otimes D_2}]_{ix,jy} = [X_a^{D_1}]_{ij}\delta_{xy} + \delta_{ij}[X_a^{D_2}]_{xy}. \tag{45}$$

Clearly we can form the direct product of an arbitrary number of representations.

Q. You know how the ket and the bra transform under the group element $\exp(i\alpha_a X_a)$. Differentiate with respect to α and divide by i to show that the action of X_a on the ket

$|i\rangle$ corresponds to the action of $-X_a$ on the bra $\langle i|$, and the commutator $[X_a, \mathcal{O}]$ for the operator $\mathcal{O}$.

Q. If $[A, B] = B$, calculate $\exp(i\alpha A) B \exp(-i\alpha A)$.

Q. Show that the structure constants indeed form a representation of the group. Use the fact that they are antisymmetric in all their indices.

Q. Show that the adjoint representation of $SU(n)$ is real.

Q. Show that if a matrix M commutes with all the generators of an irreducible representation of a Lie algebra, M must be a multiple of the unit matrix. This is the *Schur's lemma* for continuous groups. (Hint: assume M is hermitian and diagonalise it.)

6 SU(2)

$SU(2)$ is the simplest special unitary group; it is the group of all unitary unimodular 2×2 matrices. In fact, any $n \times n$ unitary unimodular matrix U (*i.e.*, $UU^\dagger = U^\dagger U = \mathbf{1}$, $\det U = 1$) can be written in terms of a hermitian matrix H:

$$U = \exp(iH). \tag{46}$$

From the identity $\det(e^A) = \exp(Tr\ A)$ [4], it follows that H must be traceless. One can write only $n^2 - 1$ linearly independent traceless hermitian matrices (can you prove that?), so $SU(n)$ will have $n^2 - 1$ generators. Among them, only $n - 1$ are diagonal. This number (of diagonal generators) is called the *rank* of the group; thus $SU(2)$ is a rank-1 group, $SU(3)$ is rank-2.

Any element of $SU(n)$ can be written as

$$U = \exp\left\{ i \sum_{a=1}^{n^2-1} \alpha_a X_a \right\} \tag{47}$$

where α_as are group parameters and X_as are the generators of the group.

For $SU(2)$, there are only three α_as and three hermitian generators. They are conveniently taken to be the three Pauli matrices divided by 2:

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{48}$$

The factor of 2 is just to make the whole thing consistent with eq. (30) with the structure constants identified with the Levi-Civita symbols — note that it is completely antisymmetric. Another reason is to make the whole thing work for spin-1/2 particles; the diagonal generator X_3 has $1/2$ and $-1/2$ as its diagonal elements.

Here one must carefully note a point. In the case discussed above, the generators themselves are $n \times n$ matrices, though they are not unimodular, and hence not members of

[4] This identity is true for hermitian matrices which can be diagonalized by unitary transformations. Suppose A is hermitian, then $UAU^\dagger$ can be diagonal if U is suitably chosen. Expanding e^A, we can say that $Ue^AU^\dagger = e^{A_d}$ where A_d is diagonal. So $\det A = \det(Ue^AU^\dagger) = \det[\exp(UAU^\dagger)] = \det[\exp(A_d)] = \det[e^{a_1}, e^{a_2}, \cdots] = \prod e^{a_i} = \exp(\sum a_i) = \exp(Tr\ A)$. We have used the fact that $\det(ABC) = \det(A)\det(B)\det(C)$ and $\det(U)\det(U^\dagger) = 1$.

$SU(n)$. However, this may not always be the case. For example, the generators in the adjoint representation are $n^2-1 \times n^2-1$ matrices. The representation where the generators are themselves of the same dimensionality as the members of the group they generate is called the *fundamental representation*. This is the lowest dimensional *nontrivial* representation of the group. We generally denote it by $\mathbf{n}$ for an $SU(n)$ group. We have seen that if X_as generate a group, so also do $-X_a^*$s. the latter representation is denoted by $\overline{\mathbf{n}}$ and is called the conjugate representation of $\mathbf{n}$, but it is equally fundamental. For $SU(2)$ only, $\mathbf{2}$ and $\overline{\mathbf{2}}$ are equivalent; thus it is a real representation. This can be verified from the relationships $\sigma_2\sigma_1\sigma_2^{-1} = -\sigma_1$, $\sigma_2\sigma_2\sigma_2^{-1} = \sigma_2$, $\sigma_2\sigma_3\sigma_2^{-1} = -\sigma_3$. For $SU(3)$ (and higher $SU(n)$s), the fundamental representation is complex, and is not equivalent to its conjugate.

Let us now construct the representations of $SU(2)$. This is so much analogous to the angular momentum algebra that we present it in a rather sketchy way. Indeed, the angular momentum algebra is nothing but the $SU(2)$ algebra; the spin-1/2 representation is, as we will see shortly, the fundamental representation, and there are higher dimensional representations for higher angular momentum systems. To make the symbols familiar to you, let us call our generators J_a instead of X_a.

We first define an operator
$$J^2 = J_iJ_i. \tag{49}$$
For the fundamental representation, $J^2 = \frac{3}{4}\mathbf{1} = \frac{1}{2}(\frac{1}{2}+1)\mathbf{1}$. Such operators which commute with all generators of a given group is called a *Casimir operator*. J_iJ_i, the sum of all generators squared, is the only quadratic (*i.e.*, second power of the generators) Casimir for any $SU(n)$ group. For $SU(3)$, we will encounter cubic Casimirs. The number of independent Casimir operators (J^4 is not an independent one) is equal to the rank of the $SU(n)$ group. Anyway, we have
$$[J^2, J_i] = 0 \tag{50}$$
for all i [5]. We define the raising and lowering operators
$$J_\pm = J_1 \pm iJ_2 \tag{51}$$
so that
$$J^2 = \tfrac{1}{2}(J_+J_- + J_-J_+) + J_3^2 \tag{52}$$
and
$$[J_+, J_-] = 2J_3, \quad [J_\pm, J_3] = \mp J_\pm. \tag{53}$$
Consider an eigenstate $|\lambda m\rangle$ of J^2 and J_3:
$$J^2|\lambda m\rangle = \lambda|\lambda m\rangle, \quad J_3|\lambda m\rangle = m|\lambda m\rangle. \tag{54}$$
It is trivial to show that the states $J_\pm|\lambda m\rangle$ are eigenstates of J^2 with eigenvalue λ and of J_3 with eigenvalues $m \pm 1$, so that we must have
$$J_\pm|\lambda m\rangle = C_\pm(\lambda, m)|\lambda m \pm 1\rangle \tag{55}$$
where the $C_\pm(\lambda, m)$ are constants to be determined later. Since $J^2 - J_3^2 \geq 0$, values of m for a given λ are bounded:
$$\lambda - m^2 \geq 0. \tag{56}$$

[5] This is a general result for any $SU(n)$ group: $[X^2, X_j] = 0$ for all j. To see this, write X^2 as X_iX_i and use the rule for the commutator $[AB, C]$. Remember that the structure constants are completely antisymmetric. Complete the proof.

Let j be the largest value of m, so that $J_+|\lambda j\rangle = 0$. Then

$$0 = J_- J_+ |\lambda j\rangle = (J^2 - J_3^2 - J_3)|\lambda j\rangle = (\lambda - j^2 - j)|\lambda m\rangle, \tag{57}$$

or $\lambda = j(j+1)$. Similarly, if j' is the smallest value of m, $\lambda = j'(j'-1)$. This gives $j(j+1) = j'(j'-1)$, whose only valid solution is $j = -j'$ (the other solution, $j' = j+1$, violates the assumption that $|\lambda j\rangle$ is the highest ket). Since J_- lowers the value of m by one unit, $j - j' = 2j$ must be an integer, or, in other words, j must be an integer or a half-integer.

It is left to you as an exercise to show that

$$\begin{aligned} C_+|\lambda m\rangle &= [(j-m)(j+m+1)]^{1/2}, \\ C_-|\lambda m\rangle &= [(j+m)(j-m+1)]^{1/2}. \end{aligned} \tag{58}$$

These states $|jm\rangle$ with $m = j, j-1, \cdots -j+1, -j$ form the basis of an $SU(2)$ irreducible representation, characterised by j. The dimension of the representation is $2j+1$ (thus, representations which are not fundamental can very well be irreducible). The representation matrices can be worked out from the relations

$$\begin{aligned} J_3|jm\rangle &= m|jm\rangle, \\ J_\pm|jm\rangle &= [(j\mp m)(j\pm m+1)]^{1/2}|j\ m\pm 1\rangle. \end{aligned} \tag{59}$$

For the fundamental representation, $j = 1/2$, $m = \pm 1/2$. Let us denote

$$|1/2, 1/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1/2, -1/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{60}$$

so that

$$J_3 = \tfrac{1}{2}\sigma_3, \quad J_1 = \tfrac{1}{2}\sigma_1, \quad J_2 = \tfrac{1}{2}\sigma_2 \tag{61}$$

which follows from

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{62}$$

This is the fundamental representation since the generators themselves are 2×2 matrices.

For $j = 1$, $m = \pm 1, 0$. We denote

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{63}$$

so that

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{64}$$

From $J_+|1,1\rangle = 0$, $J_+|1,0\rangle = \sqrt{2}|1,1\rangle$, $J_+|1,-1\rangle = \sqrt{2}|1,0\rangle$, we have

$$J_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}. \tag{65}$$

The hermitian conjugate is J_-. Thus,

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \tag{66}$$

This is the adjoint representation of $SU(2)$ — the dimension is 3×3 [6].

Q. Check that the generators in the adjoint representation satisfy the $SU(2)$ algebra. Find the similarity transformation matrix with which you could show that the adjoint representation is real.

6.1 Product Representation of SU(2)

In particle physics, we often face the situation where two or more particles combine to form a system (meson, baryon, atomic nuclei, two-electron systems, etc.). Under a particular symmetry group, one transforms as **n** and the other transforms as **m**. How does the total system transform?

Let us be more specific. Suppose we have two spin-1/2 particles; thus, both of them transform as **2** of $SU(2)$ (we often speak loosely as two **2**s of $SU(2)$). How does the combination transform? In other words, what may be the possible spin values of the combination? Here the answer is trivial: the total spin can be 0 or 1. Let us try to establish this from our knowledge of group theory.

Let r denote the first particle and s the second; r_1 and r_2 are the spin projection $+1/2$ and the spin projection $-1/2$ components (in common parlance, spin-up and spin-down states respectively), and so for s. Under $SU(2)$, they transform as

$$r'_i = U(\vec{\alpha})_{ij} r_j, \quad s'_m = U(\vec{\alpha})_{mn} s_n, \tag{67}$$

where $U(\vec{\alpha}) = \exp(i\alpha_a J_a)$, $J_a = \sigma_a/2$. Possible values of i, j, m, n depend on the representations the states are in. The product representation will transform as

$$(r'_i s'_m) = U(\vec{\alpha})_{ij} U(\vec{\alpha})_{mn} (r_j s_n) \equiv D(\vec{\alpha})_{im,jn} (r_j s_n). \tag{68}$$

Generally $D(\vec{\alpha})$ is reducible. To see what it decomposes into, take $\vec{\alpha} \ll 1$ so that we can directly work with the generators:

$$r'_i = (1 + i\alpha_a J_a^{(1)})_{ij} r_j, \quad s'_m = (1 + i\alpha_a J_a^{(2)})_{mn} s_n. \tag{69}$$

Here $J_a^{(1)}$ acts only on the first particle and does not affect the second one; similarly for $J_a^{(2)}$. The total angular momentum is

$$\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}. \tag{70}$$

The combination with largest value of J_3 ($J_3 = 1$) is $r_1 s_1$. What is the total $\mathbf{J}$ of this combination? Apply $\mathbf{J}^2$ on this combination:

$$\begin{aligned} \mathbf{J}^2 |r_1 s_1\rangle &= [(\mathbf{J}^{(1)})^2 + (\mathbf{J}^{(2)})^2 + 2\mathbf{J}^{(1)}.\mathbf{J}^{(2)}] |r_1 s_1\rangle \\ &= [(\mathbf{J}^{(1)})^2 + (\mathbf{J}^{(2)})^2 + J_+^{(1)} J_-^{(2)} + J_-^{(1)} J_+^{(2)} + 2J_3^{(1)} J_3^{(2)}] |r_1 s_1\rangle \end{aligned} \tag{71}$$

to find that

$$\mathbf{J}^2 |r_1 s_1\rangle = 2 |r_1 s_1\rangle, \tag{72}$$

so that the total angular momentum of this state is 1. We thus identify it with $|1, 1\rangle$. Apply lowering operators sucessively to find out

$$|1, 0\rangle = \frac{1}{\sqrt{2}} |r_1 s_2 + r_2 s_1\rangle, \quad |1, -1\rangle = |r_2 s_2\rangle. \tag{73}$$

[6] This is not the same representation that you get from $X_a = -i\epsilon_{abc}$, the rule for constructing the adjoint representation. But they are equivalent, so there's no problem.

The remaining state $|0,0\rangle$ must be orthogonal to $|1,0\rangle$:

$$|0,0\rangle = \frac{1}{\sqrt{2}}|r_1 s_2 - r_2 s_1\rangle. \tag{74}$$

This is antisymmetric under $1 \leftrightarrow 2$, while the earlier three states are symmetric. Thus we say, in group theoretic language,

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1} \tag{75}$$

which means that two spin-1/2 particles, which are in **2** of $SU(2)$, combine to give a triplet of $SU(2)$ (spin-1) and a singlet of $SU(2)$ (spin-0). The total number of states must remain the same. It also tells us that the representation matrices D of the combined representation is reducible into two blocks, one of dimension 3×3 and the other of dimension 1×1.

The fundamental representation of $SU(2)$ is a doublet or **2**, but it can have any n dimensional representation where $n \geq 2$. To ascertain this, just think of a system with angular momentum $(n-1)/2$; the representation must be n-dimensional. This is, however, not true for higher SU groups.

6.2 Weight and Root: A Pictorial Way of Composition

Any state of $SU(2)$ can be specified by only one quantum number, the entries in the diagonal generator X_3, provided we know the eigenvalue of X^2. The entries of X_3, *i.e.*, the eigenvalues, are called the *weight*s. Thus, the fundamental representation has only two weights: $+1/2$ and $-1/2$. It is customary to arrange the members of a multiplet by starting from the highest weight state and going downwards. For higher SU groups, there are more than one diagonal generators, and so each state is to be specified by $n-1$ weights. Again, arrangement of the members of the multiplet follows the same rule; if two states happen to have the same weight for the first diagonal generator, then we take help of the second diagonal generator to remove the degeneracy. An example will be provided when we discuss the $SU(3)$ group.

Any representation of $SU(2)$ can be shown pictorially by a line, the X_3 axis, with the states at their respective weights. Thus, for the adjoint representation, there will be three states at $+1, 0$ and -1 respectively. The vectors that take one weight to the other are called *root*s. For $SU(2)$, they are identical with the raising and the lowering operators. They will all have magnitude 1, but can point either in the raising or in the lowering direction.

Take an isospin doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$. Under an infinitesimal transformation in the isospin space, they are transformed as

$$\begin{pmatrix} p \\ n \end{pmatrix} \longrightarrow \begin{pmatrix} p' \\ n' \end{pmatrix} = [1 + i\theta\sigma_2] \begin{pmatrix} p \\ n \end{pmatrix}. \tag{76}$$

What about the electric charges of p' and n'? Well, that is an irrelevant question, since we are talking about an isospin group whose members have different electric charge and hence one must entertain the possibility of transformations that change the charge. If p and n are states of definite electric charge, p' and n' are not, but what matters is the expectation value of any operator with respect to these states. However, the main point lies elsewhere. What happens if we take the charge conjugated states? Apply the charge conjugation operator C on both sides of eq. (76) to get

$$\begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{p}' \\ \bar{n}' \end{pmatrix} = [1 + i\theta\sigma_2] \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix}. \tag{77}$$

But isospin projection is an additive quantum number, and hence it is $-1/2$ and $+1/2$ for $\bar{p}$ and $\bar{n}$ respectively. Thus, the charge conjugated states are not arranged in descending order of their weight. But we can interchange the two states; only the transformation matrix will be $[1 - i\theta\sigma_2]$. So, are $\mathbf{2}$ and $\bar{\mathbf{2}}$ different? Again, this is only illusory, as we can put an extra minus sign in front of any member of the charge conjugated doublet:

$$\begin{pmatrix} -\bar{n} \\ \bar{p} \end{pmatrix} \longrightarrow \begin{pmatrix} -\bar{n}' \\ \bar{p}' \end{pmatrix} = [1 + i\theta\sigma_2] \begin{pmatrix} -\bar{n} \\ \bar{p} \end{pmatrix}. \tag{78}$$

Thus, $\mathbf{2}$ and $\bar{\mathbf{2}}$ transform in the same way (that is what we mean by a real representation), provided we define the antiparticle doublet as shown above. The treatment is identical for the (u, d) quark doublet, and this gives us the composition of the π mesons, a triplet under $SU(2)$ of isospin:

$$\pi^+ \equiv -u\bar{d}; \quad \pi^0 \equiv \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right); \quad \pi^- \equiv d\bar{u}. \tag{79}$$

There is a normalisation factor for π^0, but the relative minus sign is important.

Getting a composite representation is easy. The steps are as follows:

- Suppose you want to combine two representations of dimensionality n_1 and n_2, *i.e.*, of multiplicity $2n_1 + 1$ and $2n_2 + 1$ respectively. Draw the weight diagram (the straight line indicating the position of the weights) of the first representation and indicate the position of the weights.

- Find the centre of mass of the second weight diagram. Put it exactly on top of the weights of the first representation. If the second representation has $2n_2 + 1$ states and you put it once on all $2n_1 + 1$ states of the first representation, the total number of states thus generated is $(2n_1 + 1)(2n_2 + 1)$. What you have done is nothing but the generation of a direct product Hilbert space. This is the number of states in the combined representation. The combined representation is reducible and our task is to get the irreducible representations.

- Start from the highest weight state of the combined representation. This must be unique. Its value will be $n_1 + n_2$; tick off all the weights of this representation, ending at the lowest weight $-n_1 - n_2$. Remember that the actual state of a given weight in a multiplet should be a linear combination of all states of that particular weight that you got by direct product (unless this state is unique). Pictorially, of course, you do not perform the operation of linear combination.

- Repeat this step till you are left with nothing. Remember that a solitary weight at $X_3 = 0$ represents a singlet. If you like, check that the sum of the number of states in all the irreducible representations is $(2n_1 + 1)(2n_2 + 1)$.

This seems to be like cracking a nut with a sledgehammer, but we will see soon what happens for $SU(3)$.

Q. Verify the following decompositions:

$$\begin{aligned} \mathbf{3} \otimes \mathbf{2} &= \mathbf{4} \oplus \mathbf{2} \\ \mathbf{4} \otimes \mathbf{2} &= \mathbf{5} \oplus \mathbf{3} \\ \mathbf{3} \otimes \mathbf{3} &= \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1} \\ \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} &= \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2} \\ \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{3} &= \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{1}. \end{aligned} \tag{80}$$

Can you find out a shortcut way to check these decompositions from angular momentum addition rules?

Q. Suppose you want to combine two states with angular momentum j_1 and j_2. The dimensionality of the reducible Hilbert space is obviously $(2j_1+1)(2j_2+1)$. Show that the dimensionalities of all the irreducible representations that you get add up to this.

6.3 SU(2) and SO(3)

The group $SU(2)$ is the group of angular momentum, *i.e.*, it has a close link with rotation. Why $SU(2)$ and not our more familiar rotation group $SO(3)$? The answer is that $SU(2)$ can take into account half-integer spins and hence fermions, whereas $SO(3)$ is good only for integer spin states. Let us clarify this statement.

Any element of $SU(2)$ can be written as $\exp(i\alpha_a X_a)$. If we are in the fundamental representation, $X_a = \frac{1}{2}\sigma_a$. Thus, *e.g.*, the element with $\alpha_1 = \alpha_3 = 0$ can be written as

$$\exp\left(i\frac{\alpha_2}{2}\sigma_2\right) = \cos\frac{\alpha_2}{2} + i\sigma_2\sin\frac{\alpha_2}{2} = \begin{pmatrix} \cos\frac{\alpha_2}{2} & \sin\frac{\alpha_2}{2} \\ -\sin\frac{\alpha_2}{2} & \cos\frac{\alpha_2}{2} \end{pmatrix} \tag{81}$$

(prove this!), which is nothing but the ordinary rotation matrix, but with half-angles. For $\alpha_2 = 2\pi$, the group element is -1 and not 1! Thus, a rotation of 2π acting on the spin-1/2 wavefunctions will bring an extra minus sign: $\psi \to -\psi$. This is something we do not expect for ordinary rotations: a rotation of 2π should bring the system back to the original configuration. Here, for $SU(2)$, you need a rotation of 4π to get back the same wavefunction.

The wavefunctions which pick up an extra minus sign under 2π rotation are called *spinors*. They are multicomponent objects, since the multiplicity under spin has to be incorporated (this will be elaborated when we discuss Lorentz groups). It is also intuitively clear that for every two rotations of $SO(3)$, there is only one rotation of $SU(2)$ (*e.g.*, rotations by 2π and 4π of $SO(3)$ correspond to a rotation by 4π of $SU(2)$; in general, $(\theta, 2\pi+\theta)_{SO(3)} \to 2\theta_{SU(2)}$.). This is why $SU(2)$ is called the covering group of $SO(3)$: there is a homomorphism of 2-to-1 between these groups.

To prove homomorphism between Lie groups, we must show that they follow the same algebra (for the discrete groups, we checked the multiplication tables). We will take the generators of $SO(3)$ to be those responsible for rotations about x, y, and z-axes. This is not the Eulerian choice, but so long as we take independent rotations, all are equally good.

For an infinitesimal rotation by θ about the z-axis, we have

$$x' = x + y\theta, \quad y' = -x\theta + y, \quad z' = z, \tag{82}$$

so that a vector $\mathbf{r} = (x, y, z)$ transforms to another vector $\mathbf{r}' = (x', y', z')$ according to

$$\mathbf{r}' = (1 + i\theta J_3)\mathbf{r}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{83}$$

Similarly we can construct J_1 and J_2. It is left for you to show that $[J_i, J_j] = i\epsilon_{ijk}J_k$, so that they satisfy the same Lie algebra as $SU(2)$. You should also show that in the fundamental representation, $J^2 = J_iJ_i = 2$, so this describes systems with angular momentum one (as $j(j+1) = 2$).

6.4 Guiding Principle to Write the Lagrangian

Any quantum mechanical system should contain wavefunctions that transform nontrivially under some continuous group transformation. What about the total Lagrangian, or the Lagrangian density, or the integrated action? Before proceeding further, let us just state the basic principle to write any Lagrangian:

> *The Lagrangian may contain fields which transform non-trivially under a given group, but the whole Lagrangian must be a singlet (*i.e, *transform as* **1***) under the group.*

This is true if the fields transform non-trivially under a number of different groups. Also note that for groups with additive quantum numbers like electric charge or hypercharge, the singlet is generally denoted by **0** and not **1**.

What does this mean? If you transform some field in a non-trivial way, some other field will transform in such a way as to cancel that effect, and the whole Lagrangian is a *scalar* in the group space. This is analogous to why the Lagrangian, or the Hamiltonian for that matter, must be a scalar in ordinary 3-dimensional space, and why it must be a Lorentz scalar in 4-dimensional space-time. After all, a 3-dimensional rotation is the group $SO(3)$; a 4-dimensional boost is a part of what is known as a *Lorentz group*.

Electromagnetism is a $U(1)$ theory; $U(1)$ is the group of Maxwellian gauge transformation, which is nothing but a phase transformation. Any field which is a singlet under $U(1)$ has a $U(1)$ quantum number zero; this quantum number may be identified with the ordinary charge of $U(1)_{em}$. Consider the Maxwell Lagrangian

$$\mathcal{L} = e\bar{\psi}\gamma^\mu\psi A_\mu \tag{84}$$

where ψ is the electron spinor. It has a charge -1, but $\bar{\psi}$, which refers to a positron, has charge $+1$, and A_μ, the photon field, has charge zero; so the total $U(1)$ quantum number is zero and $\mathcal{L}$ is a singlet under $U(1)_{em}$ — in other words, electric charge is conserved under electromagnetic interaction.

7 SU(3)

The next higher SU group is $SU(3)$. As far as I know, the application of this group is limited only to particle physics, but there it has at least two very important applications. This is not a place to discuss them; suffice it to say that this is the group for strong interaction, as well as the group for the quark model of Gell-Mann that led to the famous eightfold way.

$SU(3)$ has eight generators, and two of them can be simultaneously diagonalised at most; this is a rank-2 group. The generator matrices are conventionally written as $X_a = \frac{1}{2}\lambda_a$, a running from 1 to 8. The Gell-Mann matrices λ_a are as follows:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{85}$$

Note that all matrices are traceless and hermitian, and just like the fundamental representation of $SU(2)$, the generators are diagonalised according to $Tr(X_i X_j) = \frac{1}{2}\delta_{ij}$.

The fundamental representation **3** is complex, *i.e.*, inequivalent to $\bar{\mathbf{3}}$. The states are labeled by the weights, the eigenvalues corresponding to X_3 and X_8. Note that λ_1, λ_2, and λ_3 constitute an $SU(2)$ in the 1-2 block.

7.1 Example: Quark Model and the Eightfold Way

For concreteness, let us denote the three states of **3** by u, d, and s respectively, with orthonormal eigenvectors:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{86}$$

These states are labeled by the third component of isospin, which is nothing but the entries of X_3, and a quantum number called *strangeness*, which is a scaled variant of X_8. Strong hypercharge is defined as $2X_8/\sqrt{3}$, and subtracting baryon number (1/3 for all quarks) we get what is known as the strangeness:

$$S = \frac{2}{\sqrt{3}} X_8 - \frac{1}{3}. \tag{87}$$

Thus, for the three quarks, the I_3-S assignments are as follows:

$$u = \left(\frac{1}{2}, 0\right), \quad d = \left(-\frac{1}{2}, 0\right), \quad s = (0, -1). \tag{88}$$

The weight diagram is shown in fig. 1. For the antiquarks, which live in $\bar{\mathbf{3}}$, the assignments are just opposite. What happens when we combine **3** and $\bar{\mathbf{3}}$? From a physics point of view, that is a combination of a quark and an antiquark, and should give some mesons. The question is: how do these mesons transform under $SU(3)$?

We follow the same principle of composition: take the antiquark triangle and place its centre of mass on the three vertices of the quark triangle. The resulting regular hexagon is shown in fig. 2. There are six states at the corners of the hexagon, and three in the middle. The names of the mesons are the standard ones used by particle physicists. There are a few points to note.

- If we worked with only u and d quarks, that would have been an $SU(2)$ doublet of isospin, and so among the composites, I should have a triplet. Experimentally, the middle member of the triplet happens to be π^0, see eq. (79) for the explicit wavefunctions. An $SU(2)$ isospin transformation can be performed by the interchange of u and d, *i.e.*, a shift along the horizontal line (that is obvious, S cannot change, since both quarks have zero strangeness). A look at fig. 2 immediately tells you that there are two more $SU(2)$ doublets: K^0 and K^+, and their antiparticles, $\overline{K^0}$ and K^-.

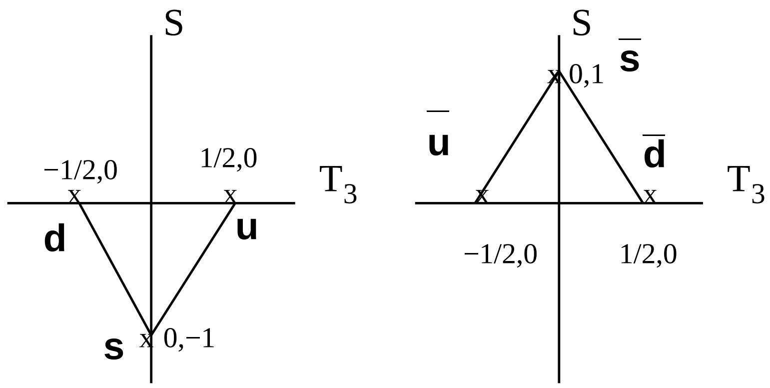

Figure 1: $\mathbf{3}$ *and* $\bar{\mathbf{3}}$ *of* $SU(3)$*. Note that due to a scaling of* X_8 *to* S*, the centre of mass of the two triangles are not the same. This would have been the case if we drew the triangles in* X_3*-*X_8 *plane.*

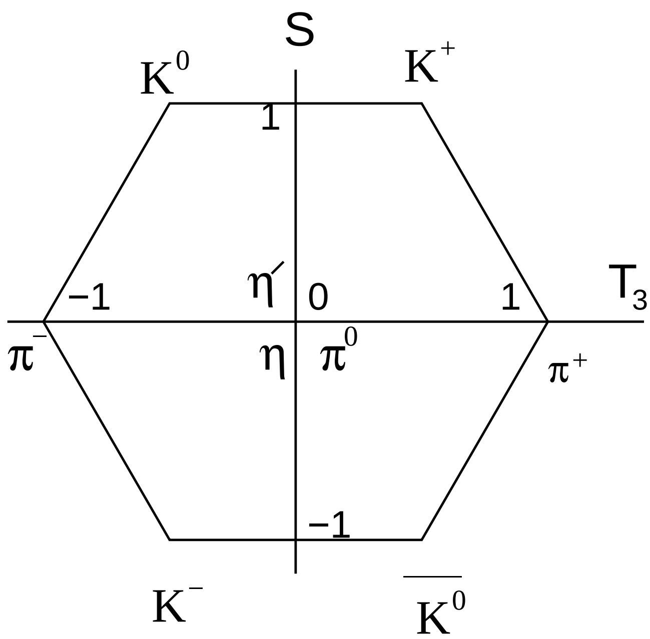

Figure 2: *Composition of* $\mathbf{3}$ *and* $\bar{\mathbf{3}}$ *of* $SU(3)$.

- The roots of $SU(3)$ take one weight to the other. So they should act along the sides of the hexagon, and all members at the sides must belong to the same multiplet. By the isospin argument, π^0 should also belong to the same multiplet. So what are the irreducible representations: $\mathbf{7} \oplus \mathbf{2}$, or $\mathbf{7} \oplus \mathbf{1} \oplus \mathbf{1}$, or what? The answer is that for any $SU(n)$, $n \otimes \bar{n} = n^2 - 1 \oplus 1$. (How do we know this, at least for $SU(3)$? We will see soon how to have an algorithm for getting the irreducible representations.) So the decomposition is $\mathbf{8} \oplus \mathbf{1}$. The singlet is one which does not transform under $SU(3)$, and hence must be symmetric in the three quarks and antiquarks. This state is labeled as $\eta' = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right)$. From the T_3 and S quantum numbers, it is one of the states at the middle of the hexagon. The third state must be an orthogonal one to this and π^0, and can be obtained by the standard Gram-Schmidt method: $\eta = \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right)$ [7].

[7] I have cheated you a bit. The actual singlet and octet states are not the physical states η' and η; rather, the latter are a combination of the former states. However, the mixing angle is small, so for the group theory

- These nine lowest mass meson states can be divided into an octet and a singlet. There are three different sets of raising and lowering operators for the octet (they have no effect on the singlet state). One of them is the set of isospin raising or lowering operators, obtained from a combination of X_1 and X_2. The other two sets are called U-spin and V-spin respectively. V-spin operators are those obtained from X_4 and X_5, and can change an up quark to a strange quark and vice-versa. U-spin operators, obtained from X_6 and X_7, take a d quark to an s quark and back. Note that only U-spin does not change the charge of the quarks and hence of the mesons.

7.2 The Method of Young Tableaux

This is the most elegant method to get the irreducible representations out of a composite. The mathematical background is somewhat complicated, and needs tensor algebra, so we will skip that here. This forces us to state only the basic rules of the game. The algorithm goes like this:

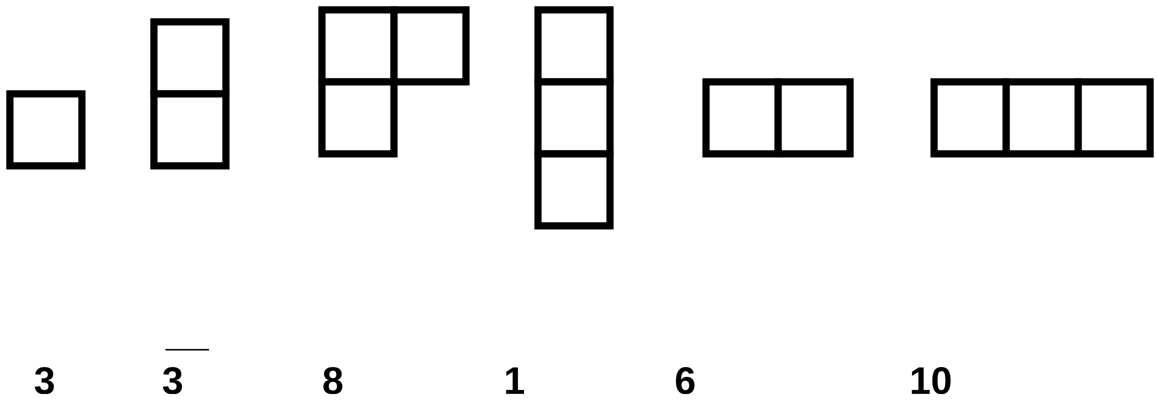

Figure 3: *Some simplest $SU(3)$ representations.*

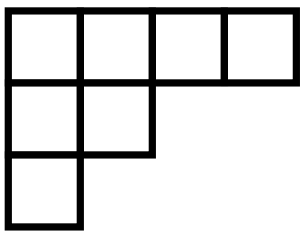

Figure 4: *A 'big' tableau, with $p = 2$ and $q = 1$.*

- Each representation of $SU(3)$ (well, there is nothing special about $SU(3)$, it can be generalised for any SU group) is pictured as a set of boxes, as shown in fig. 3 and fig. 4. This *does not* mean that a box corresponds to a wave function. Rather, the whole set, and its symmetry and antisymmetry properties, are depicted by the box pattern. The boxes must be
  - left-aligned;

purpose, if not for particle physics purpose, the assignment that I use will do.

- arranged in such a way that the number of boxes does not increase as one goes down the columns (it may decrease or remain constant);
- arranged in *not more* than three rows (this number depends on what SU group you choose; *e.g.*, it will be five for $SU(5)$).

- With such a pattern, let the difference between the number of boxes in the first and the second rows be p, and the difference between the second and the third rows be q. The dimensionality of the representation, denoted also as (p, q), is given by

$$d = \frac{1}{2}\left[(p+1)(q+1)(p+q+2)\right]. \tag{89}$$

  Thus, a single box corresponds to $(p, q) = (1, 0)$, and hence it has dimensionality 3; this is the fundamental representation of $SU(3)$. The arrangement of two boxes in a column is $(p, q) = (0, 1)$ and hence this is $\bar{\mathbf{3}}$. The representations with $p = q$ are the real representations; setting aside $p = q = 0$, which is the singlet of dimensionality 1, the lowest possible arrangement is $(p, q) = (1, 1) \Rightarrow \mathbf{8}$, the adjoint representation. (What is the dimensionality of the representation shhown in fig. 4?)

- Three boxes in a column has $p = q = 0$. This is a singlet, and anything multiplied by unity is the same thing, so composition with a singlet is trivial. That is why this particular combination is, so to say, irrelevant; to get the dimensionality of a representation, we can add or subtract at our will as many number of three boxes in a column as we wish. In any combination, if you happen to have any such arrangement, you can easily scratch that off (unless you require them to have a feeling of what q will be).

- Boxes in a row denote a symmetric combination of wavefunctions, and boxes in a column denote antisymmetric combination. While trying a composition, one must remember that symmetrised wavefunctions cannot be antisymmetrised, and vice versa. Boxes which are not in a particular row or column can be placed anywhere. A singlet is a state where the whole wavefunction is antisymmetric, *i.e.*, it picks up a minus sign every time you interchange two states:

$$|1\rangle = \frac{1}{\sqrt{6}}\left(uds - dus + dsu - sdu + sud - usd\right). \tag{90}$$

  This is a linear combinaion of six wavefunctions; the first means that u is at position 1, d at 2, and s at 3. Interchange u and d and the whole wavefunction picks up a minus sign. The prefactor is for normalisation.

- Double counting must be avoided.

Fortunately, the most important applications of $SU(3)$ Young tableaux are the most simple ones. As an example, let us try to emulate Murray Gell-Mann in his search for the eightfold way. Fig. 5 shows the final result, but we have to go through the intermediate steps.

- Unless this is a very simple composition (like joining two **3**s), always label all the boxes of the second (*i.e.*, the one on the right hand side) tableau. Label all boxes of a row in an identical way, but different from boxes in another row. For example, label all boxes of the first row by a, and all in the second row by b (the third row never counts; a full column can always be eliminated).

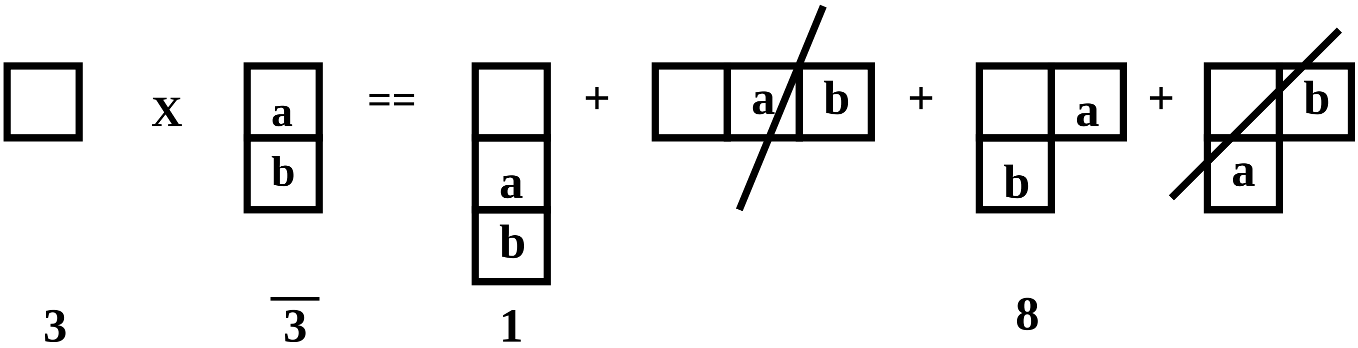

Figure 5: *An instructive but unsmart way to get* $\mathbf{3} \otimes \bar{\mathbf{3}}$.

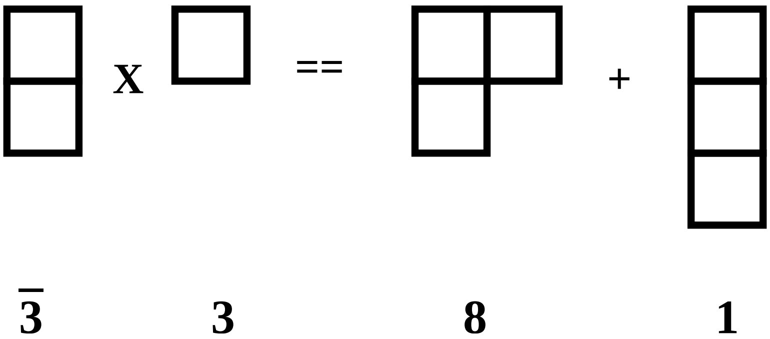

Figure 6: *The same composition can be obtained in a smarter way.*

- Take one box from the first row and join it in as many possible ways with the first tableau as possible. Of course, you have to join it only on the right hand side or below. Remember that the number of boxes in each row of this combined representation (even if this is an intermediate step) should never increase as you go down the columns. The operation gives nothing but $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$. This intermediate step, however, is not shown in fig. 5.

- Repeat this step until you finish of all the boxes in the first row. Then start the game with the boxes in the second row. Here you must be a little careful. Naively, you would have obtained all the four combinations shown in fig. 5. But two of them are not allowed: boxes a and b are in a column, *i.e.*, in an antisymmetric combination, and cannot be symmetrised, *i.e.*, put in a row. Also, in no tableaux one should have b sitting above a (they can be in the same row, if initially they belong to different columns and hence without any antisymmetry property). This is nothing but avoiding double counting. Thus, two of the boxes are ruled out, and we get the required decomposition.

- However, a smarter way is shown in fig. 6; I can always take $\mathbf{3}$ as my second box. Then only two tableaux appear and there is no question of any double counting!

- As a further example, let us consider the combination of three quarks: $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$. This is shown in fig. 7. The process is a two-step one; first we combine two $\mathbf{3}$s, and then combine the result with the last $\mathbf{3}$. Mathematically, we write

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{6} \oplus \bar{\mathbf{3}}) \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{91}$$

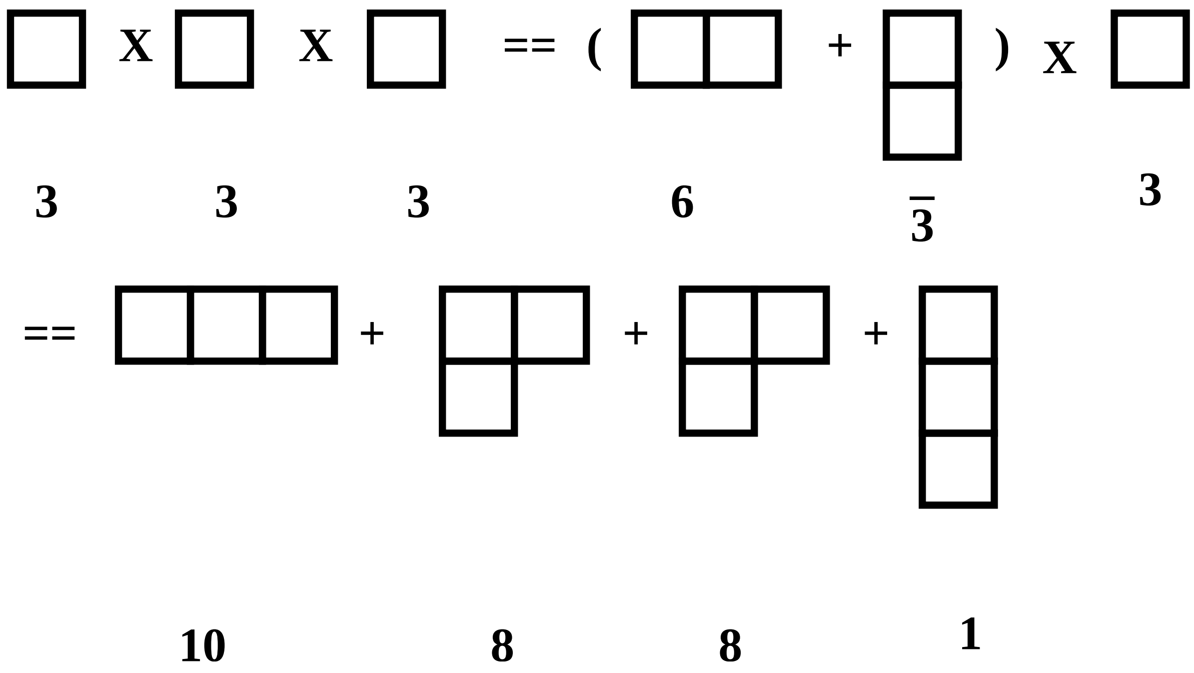

Figure 7: *Combination of three* **3**s*. In the quark model, this gives the baryons.*

The 3×3 λ matrices satisfy the following relations:

$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c, \quad \{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab}\mathbf{1} + 2d_{abc}\lambda_c, \quad Tr(\lambda_a\lambda_b) = 2\delta_{ab}. \tag{92}$$

The completeness relation is

$$(\lambda_a)_{ij}(\lambda_b)_{kl} = -\frac{2}{3}\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{jk} \tag{93}$$

where $i, j, k, l = 1, 2, 3$. The quadratic and cubic Casimir operators are defined as

$$C_2(R)\mathbf{1} \equiv \sum_{a=1}^{8}(X_a)^2(R), \quad C_3(R)\mathbf{1} \equiv \sum_{a,b,c=1}^{8} d_{abc}X_a(R)X_b(R)X_c(R), \tag{94}$$

where R is the dimension of the representation (3 for fundamental) and $X_a = \lambda_a/2$. It can be shown that

$$\begin{aligned} C_2(p,q) &= \frac{1}{3}\left[p^2 + pq + q^2 + 3p + 3q\right], \\ C_3(p,q) &= \frac{1}{18}(p-q)(2p+q+3)(2q+p+3). \end{aligned} \tag{95}$$

This gives, *e.g.*, $C_2(3) = 4/3$, $C_2(8) = 3$.

Now, some important relations. From eq. (93), we have

$$(\lambda_a)_{ij}(\lambda_a)_{jl} = \left(-\frac{2}{3} + 2 \times 3\right)\delta_{il} = 4C_2(3)\delta_{il}. \tag{96}$$

From the definition of the adjoint representation $(X_a)_{bc} = -if_{abc}$,

$$f_{acd}f_{bcd} = C_2(8)\delta_{ab} = 3\delta_{ab}. \tag{97}$$

Q. Perform the topnotch exercise: $\mathbf{8} \otimes \mathbf{8} = \mathbf{27} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{8} + \mathbf{8} + \mathbf{1}$. Be careful to avoid double counting. Also preserve the symmetry and the antisymmetry properties.

Q. Show that $f_{abc}\lambda_b\lambda_c = iC_2(8)\lambda_a$. (*Hint*: write $\lambda_b\lambda_c = \frac{1}{2}[\lambda_b, \lambda_c]$ since f_{abc} is completely antisymmetric, and then use eq. (97).)

Q. Justify the steps:

$$\begin{aligned} \lambda_b\lambda_a\lambda_b &= \frac{1}{2}\left(\lambda_b[\lambda_a, \lambda_b] - [\lambda_a, \lambda_b]\lambda_b + \lambda_b\lambda_b\lambda_a + \lambda_a\lambda_b\lambda_b\right) \\ &= 4C_2(3)\lambda_a + if_{abc}[\lambda_b, \lambda_c] = 4\left\{C_2(3) - \frac{1}{2}C_2(8)\right\}\lambda_a. \end{aligned} \tag{98}$$

8 Lorentz Group

Let us recall that the ordinary rotation groups $SO(2)$ and $SO(3)$ have one and three generators respectively. They are the groups for two-dimensional and three-dimensional rotations. The generator of $SO(2)$ is just σ_2, while $SO(3)$ has three generators J_1, J_2, and J_3 satisfying $[J_i, J_j] = i\epsilon_{ijk}J_k$.

Now let us consider the Lorentz transformations (LT). We take the metric to be Minkowski, with $g^{\mu\nu} = diag(+1, -1, -1, -1)$, and set $c = 1$. The three generators for rotation will be still there, but they are enlarged, *e.g.*,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{99}$$

and permutations generate J_2 and J_3. The extra row and column of zeroes remind us that the space-time is four-dimensional, (x^0, x^1, x^2, x^3), and rotations do not affect the time dimension. The J_is satisfy the same $SO(3)$ algebra as before. However, the group has three more generators, boosts along the three spatial axes, denoted by K_i. Consider a boost in the $x \equiv x^1$ direction, which can be written as

$$t' = t\cosh\phi - x\sinh\phi, \quad x' = x\cosh\phi - t\sinh\phi\,, \tag{100}$$

where $\tanh\phi = v$. For small ϕ, this becomes $t' = t - x\phi, x' = x - t\phi$, so that the generator $\mathcal{K}_1$ has the form

$$\mathcal{K}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{101}$$

(What should be the forms of $\mathcal{K}_2$ and $\mathcal{K}_3$?)

Note that a finite transformation is given by $\exp(\alpha_a\mathcal{K}_a)$; there is no i, and the αs are real, so the group is a *noncompact* one. Thus, it need not be represented by unitary matrices. However, to work out the group algebra, we should express the group operations in an identical way:

$$\Lambda(\theta_a, \alpha_a) = \exp(i\theta_aJ_a + i\alpha_aK_a)\,, \tag{102}$$

where $\mathcal{K}_a = iK_a$, and thus, for example,

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{103}$$

Now compute by brute force to verify that

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad \text{but } [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{104}$$

This last commutator is interesting: the commutator of two boosts is a rotation. In other words, if you boost some object along the x-axis, and then along the y-axis, but boost it back to the original position by first applying a reverse boost along the y-axis and then along the x-axis, you will find that the body has undergone a rotation about the z-axis! This is known as the Thomas precession.

The Lorentz group, denoted by $SO(3,1)$, is reducible [8]. This is one of its most important features. Construct the generators $J_{\pm i} = \frac{1}{2}(J_i \pm iK_i)$. Check that

$$[J_{+i}, J_{+j}] = i\epsilon_{ijk}J_{+k}, \quad [J_{-i}, J_{-j}] = i\epsilon_{ijk}J_{-k}, \quad \text{and } [J_{+i}, J_{-j}] = 0. \tag{105}$$

The algebra falls apart in two $SU(2)$s. To be very precise, $SO(3,1)$ is isomorphic to $SU(2) \otimes SU(2)$. Therefore, any representation of the Lorentz group can be written as (j_1, j_2), a pair of $SU(2)$ representations, with a dimensionality of $(2j_1+1)(2j_2+1)$. With increasing dimensionality, the representations are $(0,0)$, $(1/2,0)$ and $(0,1/2)$, $(1,0)$ and $(0,1)$, $(1/2,1/2)$, and so on. The Lorentz scalar is $(0,0)$, and the fundamental representation is the 4-dimensional Lorentz vector $(1/2,1/2)$.

Why, then, we use a four-component Dirac spinor, instead of a two-component Weyl spinor? The reason is parity: under parity operation, J_i does not change sign (angular momentum is an axial vector) but K_i does, and so $J_{\pm i} \to J_{\mp i}$. In other words, $(1/2,0) \leftrightarrow (0,1/2)$. These are known as the left-chiral and the right-chiral representations respectively. But electrodynamics is invariant under parity, so we must have some wavefunction that is a combination of two $SU(2)$ doublets. If we have some theory that violates parity, we can do with two-component spinors, and indeed under weak interaction, the two Weyl doublets behave differently.

8.1 More about Lorentz Groups

There is another way to look at the Lorentz groups. What we have just discussed is in fact only a restricted subset of the full group. A *homogeneous* Lorentz group is characterised by the coordinate transformation

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{106}$$

or, in matrix notation, $x' = \Lambda x$. This must leave the quadratic form $x^2 = x^\mu x_\mu = (x^0)^2 - (x^i)^2$ invariant. All entries in the matrix Λ are real, so the condition of invariance of the quadratic form can be written as

$$\Lambda^\mu_\nu \Lambda^\nu_\rho = \delta^\mu_\rho. \tag{107}$$

[8]Technically, what we are interested in is a subset of the full Lorentz group that is continuously connected to the identity. Sometimes called the *restricted* Lorentz group, this is denoted by $SO^+(3,1)$, but we will not be that fussy about mathematical notations.

This can be written as

$$\Lambda^\mu_\nu g_{\mu\sigma} \Lambda^\sigma_\rho = g_{\nu\rho}, \tag{108}$$

or, in other words, since $\Lambda^\mu_\nu = (\Lambda^T)^\nu_\mu$,

$$\Lambda^T g \Lambda = g, \tag{109}$$

where the superscript T denotes the transposed matrix. It follows from this equation that $det\Lambda = \pm 1$ and hence every homogeneous LT has an inverse transformation. The product of two LTs is another LT, and hence LTs form a group. Obviously, the Lorentz group has a proper subgroup: the group of 3-dimensional rotation, $SO(3)$.

Setting $\nu = \rho = 0$ in eq. (108), we get

$$(\Lambda^0_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i_0)^2 \geq 1, \tag{110}$$

so that $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$. The first set is known as *orthochronous* LT, the second is non-orthochronous. Orthochronous LTs transform a positive time-like vector into another positive time-like vector, and the set of all orthochronous LTs form a group: the orthochronous Lorentz group. (Why the non-orthochronous set is not a group?)

Again, for both the choices of Λ^0_0, $det\Lambda$ can be $+1$ or -1. The subset with $\Lambda^0_0 \geq 1$ and $det\Lambda = +1$ is the group of restricted homogeneous LTs (this is the group we were discussing in the last subsection). All elements are continuously connected to identity.

The other three subsets are obtained by adjoining the following transformations to the restricted homogeneous LT group:

$$\text{Space inversion } (x^0 \to x^0, \mathbf{x} \to -\mathbf{x}): \qquad \Lambda(i_s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\text{Time inversion } (x^0 \to -x^0, \mathbf{x} \to \mathbf{x}): \qquad \Lambda(i_t) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\text{Space} - \text{time inversion } (x^0 \to -x^0, \mathbf{x} \to -\mathbf{x}): \qquad \Lambda(i_{st}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{111}$$

These subsets are disjoint and are not continuously connected to identity.

Q. Convince yourself of the validity of eq. (108).
Q. Which of the three improper LTs is orthochronous?

8.2 The Inhomogeneous Lorentz Group

The inhomogeneous Lorentz group (also called the Poincaré group) is characterised by the transformation

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \tag{112}$$

where a^μ is a real four-vector. So this is nothing but a product transformation of a homogeneous LT and a translation. Therefore it has ten generators, six from the homogeneous Lorentz group and four from translation, which are nothing but p^μ. The algebra is enlarged by the commutators

$$[p_\mu, p_\nu] = 0, \quad [M_{\mu\nu}, p_\sigma] = i(g_{\nu\sigma}p_\mu - g_{\mu\sigma}p_\nu), \tag{113}$$

where $M_{\mu\nu}$ is the hermitian generators for homogeneous LT in x^μ-x^ν plane.

9 Epilogue

This is only a sketchy note. I have avoided detailed deductions as far as possible. You should not avoid them. You should definitely work out the problems, and think whenever you are expected to think. Remember that exploration of symmetries is what physics is mostly about. Group theory is a tool to help you, but it is not the physics itself.

For those who wish to be condensed matter physicists, the discussion on discrete groups is terribly limited here. You will have to learn much more about the crystallographic point groups, but this note may act as a good starter. Be careful about the notation: I have used my own notations, which may not always tally with the standard ones. Fortunately, physics does not depend on notations, but communication skill may.

The discussion on Lie groups is also far from exhaustive. There are a number of topics I have not touched, but you can always come back and read a good text if you need them. This discussion should be able to get you started in your course on nuclear or particle physics.

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